

EQUIVARIANT COHOMOLOGY OF COHOMOGENEITY ONE ACTIONS: THE TOPOLOGICAL CASE

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ABSTRACT. We show that for any cohomogeneity one continuous action of a compact connected Lie group G on a closed topological manifold the equivariant cohomology equipped with its canonical $H^*(BG)$ -module structure is Cohen-Macaulay. The proof relies on the structure theorem for these actions recently obtained by Galaz-García and Zarei. We generalize in this way our previous result concerning smooth actions.

Let M be a closed topological manifold, G a compact connected Lie group, and $G \times M \rightarrow M$ a continuous action of cohomogeneity one, i.e., such that the orbit space M/G is one-dimensional. A complete description of such group actions has been obtained only recently by Galaz-García and Zarei in [5]. As a consequence, they were able to identify within the setup above the actions which do not fit into the smooth category. We recall that smooth cohomogeneity-one actions had been previously dealt with by Mostert in [8].

Our main goal here is to prove the following result. (The coefficient ring for cohomology is always \mathbb{R} .)

Theorem 1. *Let M and G be as above, such that the action has cohomogeneity equal to 1. Then the equivariant cohomology group $H_G^*(M)$ equipped with its canonical structure of a module over $H^*(BG)$ is Cohen-Macaulay.*

We recall that $H^*(BG)$ is a polynomial ring in several variables. Furthermore, a non-zero module over any such ring is Cohen-Macaulay if it is finitely generated and its depth is equal to its Krull dimension (see, e.g., [3, Sect. 1.5]).

In the special case when the G -action is smooth, the theorem above has been proved in [6]. We will try to keep the present paper self-contained; however, for the sake of brevity, sometimes we will prefer to simply invoke results already proved in the aforementioned work.

The structure theorem of [5, 8] says as follows. Let M and G as indicated above. Then the orbit space M/G is homeomorphic to either the circle S^1 or the interval $[0, 1]$. In the first case there is just one orbit type, say (H) , and M is a G/H -bundle over S^1 . If $M/G = [0, 1]$ then M is determined by a *group diagram*. This is of the form (G, H, K^-, K^+) here H, K^-, K^+ are closed subgroups of G such that $H \subset K^\pm$ and each of K^+/H and K^-/H is homeomorphic to a sphere or to the Poincaré homology sphere \mathbf{P}^3 . The cohomogeneity one manifold encoded by this quadruple is

$$M = G \times_{K^-} C(K^-/H) \cup_{G/H} G \times_{K^+} C(K^+/H),$$

where $C(K^\pm/H) = (K^\pm/H \times [0, 1]) / (K^\pm/H \times \{1\} = *)$ are the cones over K^\pm/H . The G -action has three isotropy types, which are represented by H (the principal one), K^- , and K^+ , respectively.

The following result will be needed later; it is similar in spirit to [6, Proposition 3.1].

Proposition 2. *Let K be a compact Lie group, possibly non-connected, which acts transitively on the Poincaré homology sphere \mathbf{P}^3 and let $H \subset K$ be an isotropy subgroup. Then $\text{rank } H = \text{rank } K - 1$ and the canonical homomorphism $H^*(BK) \rightarrow H^*(BH)$ is surjective.*

Proof. If the K -action is effective, then, by [2], K is a rank 1 Lie group and H is a finite group, hence the ranks satisfy the required equation (note that in [2] it is assumed that K is connected: however, if this not the case, observe that the action of the identity component of K on \mathbf{P}^3 is transitive as well). Let us now assume that the kernel of the action, call it K' , is a non-trivial (normal) subgroup of K . The desired conclusion about the ranks follows by writing

$K/H = (K/K')/(H/K')$ and noticing that $\text{rank } K = \text{rank } (K/K') + \text{rank } K'$ and $\text{rank } H = \text{rank } (H/K') + \text{rank } K'$, see, e.g., [1, Ch. 9, Prop. 2 (c)]. It now remains to justify the surjectivity claim. But this is a straightforward consequence of the generalized Gysin cohomology sequence for the fibration $\mathbf{P}^3 \rightarrow BH \rightarrow BK$, see, e.g., [9, Theorem 2, Sect. 9, Ch. 5]; one takes into account that BH and BK have vanishing cohomology in odd degrees. \square

We are now in a position to prove the main result.

Proof of Theorem 1. If $M/G = S^1$ then, as already mentioned, all isotropies are conjugate to each other. The G -action on M is Cohen-Macaulay by [7, Corollary 4.3], see also [4, Corollary 1.2]. The same argument works also in the case when $M/G = [0, 1]$ and $\text{rank } H = \text{rank } K^- = \text{rank } K^+$. Let us now assume that $M/G = [0, 1]$ and $\text{rank } H \leq \text{rank } K^+ \leq \text{rank } K^-$ but $\text{rank } H < \text{rank } K^-$. Denote $b := \text{rank } K^-$. By [6, Proposition 3.1] and Proposition 2 above, $\text{rank } H = b - 1$. Consider the open G -invariant covering $M = U \cup V$ with

$$\begin{aligned} U &:= G \times_{K^-} C(K^-/H) \cup_{G/H} G \times_{K^+} (K^+/H \times [0, \epsilon)) \text{ and} \\ V &:= G \times_{K^-} (K^-/H \times [0, \epsilon)) \cup_{G/H} G \times_{K^+} C(K^+/H) \end{aligned}$$

where $\epsilon > 0$ is small. Note that U and V contain G/K^+ and G/K^- respectively as G -equivariant deformation retracts. Thus the corresponding Mayer-Vietoris sequence is

$$\cdots \longrightarrow H_G^*(M) \longrightarrow H_G^*(G/K^-) \oplus H_G^*(G/K^+) \longrightarrow H_G^*(G/H) \longrightarrow \cdots.$$

By [6, Proposition 3.1] and Proposition 2, the last map in the sequence above is surjective, hence the sequence splits:

$$(1) \quad 0 \longrightarrow H_G^*(M) \longrightarrow H^*(BK^-) \oplus H^*(BK^+) \longrightarrow H^*(BH) \longrightarrow 0.$$

If both K^-/H and K^+/H are spheres, the proof in [6, Sect. 4] goes through without any modification. Assume now that at least one of K^-/H and K^+/H is homeomorphic to \mathbf{P}^3 . Recall that $b - 1 \leq \text{rank } K^+ \leq b$.

Case 1: $\text{rank } K^+ = b$. We use the same argument as in [6, Sect. 4, Case 1].

Case 2: $\text{rank } K^+ = b - 1$. By Proposition 2, K^+/H is a sphere. Thus this time we simply need to notice that everything in [6, Sect. 4, Case 2] remains valid in this new setup. \square

As a byproduct we have proved that if M/G is a closed interval and $\text{rank } H < \max\{\text{rank } K^-, \text{rank } K^+\}$ then $H_G^*(M)$ is isomorphic to the kernel of

$$H^*(BK^-) \oplus H^*(BK^+) \rightarrow H^*(BH), (f, g) \mapsto \pi_1(f) - \pi_2(g),$$

where π_1 and π_2 are the obvious maps.

In general, the Krull dimension of $H_G^*(M)$ over the ring $H^*(BG)$ is equal to the maximal rank of the isotropy subgroups. The action is equivariantly formal, i.e.,

the module mentioned above is free, if and only if there is at least one isotropy subgroup of the same rank as G . To illustrate this principle, recall that the main difference between topological and smooth cohomogeneity-one actions is that only in the former case one can have $K^-/H = \mathbf{P}^3$ or $K^+/H = \mathbf{P}^3$; now, such an action can be equivariantly formal only if M is even dimensional (otherwise, by [6, Sect. 5.2], all isotropies would have maximal rank). A concrete situation is analyzed below.

Consider the (non-smoothable) cohomogeneity-one action described in [5, Example 2.3]. The group diagram is $(S^3 \times \mathrm{SO}(n+1), I^* \times \mathrm{SO}(n), I^* \times \mathrm{SO}(n+1), S^3 \times \mathrm{SO}(n))$, where I^* is the binary icosahedral group; the manifold acted on is $\mathbf{P}^3 * S^n$. Identify $H^*(BS^3) = \mathbb{R}[u]$, where $\deg u = 4$. The equivariant cohomology is the kernel of the map $H^*(BSO(n+1)) \oplus \mathbb{R}[u] \otimes H^*(BSO(n)) \rightarrow H^*(BSO(n))$, $(f, g) \mapsto \pi_1(f) - \pi_2(g)$, as a module over $\mathbb{R}[u] \otimes H^*(BSO(n+1))$. If n is odd then there exists $f \in \ker \pi_1$, $f \neq 0$; but then $f.(0, u) = 0$, hence the module is not free, in conformity with the previous discussion. If n is even, then π_1 is injective and the equivariant cohomology is isomorphic to $\pi_1(H^*(BSO(n+1))) + u\mathbb{R}[u] \otimes H^*(BSO(n))$, which is a subring of $\mathbb{R}[u] \otimes H^*(BSO(n))$. This module over $\mathbb{R}[u] \otimes H^*(BSO(n+1))$ is now free: a basis consists of 1 and ue , where $e \in H^*(BSO(n))$ is such that $\{1, e\}$ is a basis of $H^*(BSO(n))$ over $H^*(BSO(n+1))$.

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